

ANNIHILATORS IN \mathbb{N}^k -GRADED AND \mathbb{Z}^k -GRADED RINGS

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ABSTRACT. It has been shown by MCCOY that a right ideal of a polynomial ring with several indeterminates has a non-trivial homogeneous right annihilator of degree 0 provided its right annihilator is non-trivial to begin with. In this note, it is shown that any \mathbb{N}^k -graded ring R has a slightly weaker property: any non-trivial annihilator of a right ideal contains a homogeneous non-zero element. If R is a subring of a strongly \mathbb{Z}^k -graded ring then it is possible to find annihilators of degree 0. This generalises MCCOY's results on polynomial rings, and puts them into the framework of graded algebra.

Let G be an additive monoid with zero element 0. We say that a G -graded ring T has the *graded right MCCOY property for elements* if for all non-zero $f, g \in T$ with $f \cdot g = 0$ there exists a homogeneous element $h \in T$ of degree 0 such that $f \cdot h = 0$. We say that T has the *graded right MCCOY property for ideals* if for all right ideals $A \trianglelefteq T$ with non-trivial right annihilator A^r there exists a non-zero homogeneous element of degree 0 in A^r .

It was shown by MCCOY that a polynomial ring $T = K[X_1, X_2, \dots, X_k]$, for K a unital ring, has the graded right MCCOY property for ideals [McC57, Theorem]. Earlier, MCCOY had established that if K is unital and commutative then T has the graded right MCCOY property for elements [McC42, Theorems 2 and 3].

It is the purpose of this note to re-visit these results strictly from the point of view of graded algebra, and thus provide a different perspective on the special place polynomial rings occupy in the theory. We show that every \mathbb{N}^k -graded ring T has the *weak graded right MCCOY property for ideals* (Theorem 3.1): if the right ideal $A \trianglelefteq T$ has non-trivial right annihilator A^r there exists a non-zero homogeneous element, of possibly non-zero degree, in A^r . If T arises as the \mathbb{N}^k -graded subring of a *strongly* \mathbb{Z}^k -graded ring then T actually possesses the graded right MCCOY property for ideals (Theorem 4.2); this applies, for example, to polynomial rings.

For semi-commutative T the (weak) graded MCCOY property for ideals implies the (weak) graded MCCOY property for elements, which is recorded in Corollaries 3.2 and 5.3.

1. NOTATION AND CONVENTIONS

All rings considered have a unit element. Given a right ideal A of a ring R we denote by A^r the right annihilator of A in R , that is, the two-sided ideal $A^r = \{r \in R \mid \forall a \in A: ar = 0\}$ of R .

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Rings graded by a monoid. Given a monoid G , additively written, a G -graded ring is a ring R together with a decomposition $R = \bigoplus_{g \in G} R_g$ into ABELian groups such that $R_g R_h \subseteq R_{g+h}$ for all $g, h \in G$. Elements of R_g are called *homogeneous of degree g* ; we may say *R -homogeneous of degree g* if we want to emphasise the ring R and its grading.

Every element r of a G -graded ring R can be uniquely written as a sum

$$r = \sum_{g \in G} r_g \quad (1.1)$$

where $r_g \in R_g$, with almost all r_g zero.

Lemma 1.2. *Suppose G is an additively written monoid with zero element 0 . Let T be a G -graded ring.*

- (1) *If G is right cancellative (so that $a + c = b + c$ implies $a = b$) or left cancellative, the unit element $1 \in T$ is homogeneous of degree 0 .*
- (2) *Let $r = \sum_{g \in G} r_g \in T$ as in (1.1), and let $s \in T$ be homogeneous.*
 - (a) *If G is right cancellative, $rs = 0$ implies $r_g s = 0$ for all g .*
 - (b) *If G is left cancellative, $sr = 0$ implies $sr_g = 0$ for all g .*

Proof. To prove (1), suppose first that G is right cancellative. Write $r = 1 = \sum_{g \in G} r_g$ as in (1.1). For any homogeneous element $s \in T_h$ we find $s = 1 \cdot s = \sum_{g \in G} r_g s$, with $r_g s \in T_{g+h}$. By uniqueness of the representation, this implies $r_g s = 0$ whenever $h \neq g + h$, i.e., whenever $g \neq 0$ (recall that G is right cancellative). By distributivity, we have $r_g z = 0$ for any $z \in T$ and $g \neq 0$. Applying this to $z = 1$ yields

$$1 = z = 1 \cdot z = \sum_{g \in G} r_g z = r_0 z = r_0 \cdot 1 = r_0$$

which shows that the unit is homogeneous of degree 0 as claimed. — The left cancellative case is similar.

Part (2) is similar but easier; the point of having a right cancellative monoid, for example, is that the elements $r_g s \in R_{g+h}$ have different degrees for distinct $g \in G$ so that no annihilation of non-zero terms can occur. \square

Rings graded by \mathbb{N} . Suppose that $T = \bigoplus_{j \geq 0} T_j$ is an \mathbb{N} -graded ring. Every non-zero element $z \in T$ can be written uniquely in the form

$$z = \sum_{j=\ell}^u z_j \quad (1.3)$$

with $z_j \in T_j$, and both z_ℓ and z_u non-zero. We call the expression (1.3) the *canonical form* of z . We say that z has *lower degree ℓ* and *upper degree u* ; the quantity $a = u - \ell \geq 0$ is called the *amplitude* of z . The element z is homogeneous if and only if it is of amplitude 0 . We remark that if x has amplitude a and z is a homogeneous element, then xz has amplitude not exceeding a . Indeed, if ℓ and u denote the lower and upper degree of x , respectively, and z has degree d , then the lower degree of xz is at least $\ell + d$ while the upper degree of xz is at most $u + d$. Inequality occurs if and only if $x_\ell z = 0$ or $x_u z = 0$.

2. ANNIHILATORS IN \mathbb{N} -GRADED RINGS

Theorem 2.1. *Suppose that T is an \mathbb{N} -graded ring. Let A be a right ideal in T such that A^r , its right annihilator, contains a non-zero element y of positive amplitude a . Then A^r also contains a non-zero element z of amplitude less than a . In particular, A^r contains a non-zero homogeneous element.*

A version of this Theorem for commutative polynomial rings, referring to degree rather than amplitude, was given by FORSYTHE [For43, Theorem A].

Proof. Write y in canonical form $y = \sum_{\ell}^u y_j$; note that $a = u - \ell > 0$ by hypothesis. If $y_u \in A^r$ then $z = y_u$ has amplitude 0, and is the desired element of amplitude less than a .

Otherwise, if $y_u \notin A^r$, we can choose an element $x \in A$, with canonical form $\sum_{i=m}^n x_i$, such that $xy_u \neq 0$. If $x_i y = 0$ for all i then we must in particular have $x_i y_u = 0$ for all i by Lemma 1.2 (2)(b). Thus $xy_u = \sum_m^n x_i y_u = 0$ contradicting the choice of x . Consequently, there exists a maximal index p with $x_p y \neq 0$. But $y \in A^r$ so that

$$0 = xy = \sum_{i=m}^n x_i y = \sum_{i=m}^p x_i y .$$

It follows that $x_p y_u = 0$ so that $z = x_p y$ has amplitude less than a ; indeed, the upper degree of $x_p y$ is less than $p + u$, while the lower degree of $x_p y$ is at least $p + \ell$. — As A^r is a two-sided ideal, $z = x_p y \in A^r$ is the desired non-zero element of amplitude less than a .

The last sentence of the Theorem follows by repeated application of what we proved already, yielding non-zero elements in A^r of successively smaller amplitude. The process must stop with an element of amplitude 0. \square

Remark 2.2. The result is best possible. For let $T = K[X, Z]/\langle XZ, Z^2 \rangle$, with K a field, graded by $\deg(X) = \deg(Z) = 1$. Let A be the right ideal of all polynomials in X and Z without constant term. This ideal is annihilated by the element $Z \in T_1$, but there is no homogeneous annihilator of degree 0.

3. ANNIHILATORS IN \mathbb{N}^k -GRADED RINGS

Theorem 3.1. *Suppose that R is an \mathbb{N}^k -graded ring. Let A be a right ideal in R such that A^r , its right annihilator, is non-trivial. Then A^r contains a non-zero homogeneous element.*

Proof. We use induction on k , the case $k = 1$ being the final sentence of Theorem 2.1, applied to $T = R$.

So suppose that R is \mathbb{N}^{k+1} -graded. We let T denote the \mathbb{N} -graded ring which is identical to R as a ring, but with grading defined by the last coordinate of \mathbb{N}^{k+1} . More explicitly, denote by $\tau: \mathbb{N}^{k+1} \longrightarrow \mathbb{N}$ the composition

$$\mathbb{N}^{k+1} = \mathbb{N}^k \oplus \mathbb{N} \longrightarrow \mathbb{N} ;$$

writing $R = \bigoplus_{v \in \mathbb{N}^{k+1}} R_v$ we let $T = \bigoplus_{j=0}^{\infty} T_j$ where

$$T_j = \bigoplus_{v \in \tau^{-1}(j)} R_v .$$

Now suppose A is a right ideal of the ring $R = T$ which has non-trivial right annihilator A^r . By Theorem 2.1 we find a non-zero element $y \in A^r$ which is homogeneous of degree d as an element of T , *i.e.*, $y \in A^r \cap T_d$.

Write a general element $x \in A \subseteq T$ in canonical form with respect to the ring T (*i.e.*, with respect to the \mathbb{N} -grading),

$$x = \sum_{i=m}^n x_i ,$$

and let J denote the right ideal of $T = R$ generated by all the resulting elements x_i (letting x vary over all of A); that is, J is the smallest *graded* ideal of T containing A . — As $x_i \in T_i$ for all i , and as $y \in T_d$ is homogeneous, the equality $xy = 0$ (true since $y \in A^r$) implies $x_i y = 0$ for all i , by Lemma 1.2 (2)(a). Thus in fact $y \neq 0$ is a right annihilator of J , that is, $y \in J^r$.

Next, we let S denote the \mathbb{N}^k -graded ring which is identical to R (and T) as a ring, but with grading given by the first k coordinates of $\mathbb{N}^{k+1} = \mathbb{N}^k \oplus \mathbb{N}$. More explicitly, denote by $\sigma: \mathbb{N}^{k+1} \longrightarrow \mathbb{N}^k$ the composition

$$\mathbb{N}^{k+1} = \mathbb{N}^k \oplus \mathbb{N} \longrightarrow \mathbb{N}^k ;$$

writing $R = \bigoplus_{v \in \mathbb{N}^{k+1}} R_v$ as before we let $S = \bigoplus_{s \in \mathbb{N}^k} S_s$ where

$$S_s = \bigoplus_{v \in \sigma^{-1}(s)} R_v .$$

Recall now that $J \trianglelefteq S$ has non-trivial right annihilator as it contains the element $y \neq 0$ constructed above. By our induction hypothesis, applied to the \mathbb{N}^k -graded ring S and the ideal J , we can find a homogeneous non-zero right annihilator z of J in S , of degree $s \in \mathbb{N}^k$ say. Such an element can uniquely be written as a sum of non-zero elements

$$z = z_{v_1} + z_{v_2} + \dots + z_{v_\ell} ,$$

with $z_{v_j} \in R_{v_j}$ and $\sigma(v_j) = s$ for all j such that

$$\tau(v_1) < \tau(v_2) < \dots < \tau(v_\ell) .$$

We specifically choose z with ℓ as small as possible. If $\ell = 1$ we are done: the element $z = z_{v_1}$ is a homogeneous element of R which annihilates J , and thus annihilates A ; note that $J^r \subseteq A^r$. On the other hand, $\ell > 1$ cannot happen. Indeed, if $\ell > 1$ then $z_{v_\ell} \notin J^r$ by minimality of ℓ . This means we can find an element $x \in J$ with $xz_{v_\ell} \neq 0$. In fact, as $J \trianglelefteq T$ is a graded ideal, as remarked before, we can ensure that x is *T -homogeneous*. But then $xz = 0$ implies $xz_{v_\ell} = 0$ by Lemma 1.2 (2)(b), a contradiction. Thus we *must* have $\ell = 1$, finishing the induction. \square

Corollary 3.2. *Let R be an \mathbb{N}^k -graded ring, and let $(f_i)_{i \in I}$ be a family of elements of R . Suppose there exists a non-zero element $g \in R$ such that $f_i g = 0$ for all i . Suppose further that R is semi-commutative so that $ab = 0$ implies $arb = 0$ for all $r \in R$. Then there exists a non-zero homogeneous element $h \in R_v$, of possibly non-zero degree $v \in \mathbb{N}^k$, such that $f_i h = 0$ for all $i \in I$.*

Proof. Let $A = \langle f_i \mid i \in I \rangle$ be the right ideal generated by the elements f_i . As R is semi-commutative, g is a non-trivial annihilator of A . Hence by

Theorem 3.1 there exists a homogeneous non-zero element $h \in A^r$; this element h annihilates in particular the specified generators f_i of A . \square

4. ANNIHILATORS IN POSITIVE SUBRINGS OF HALF-STRONGLY \mathbb{Z}^k -GRADED RINGS

Let $S = \bigoplus_{v \in \mathbb{Z}^k} S_v$ now denote a unital \mathbb{Z}^k -graded ring. We write

$$S_+ = \bigoplus_{v \in \mathbb{N}^k} S_v$$

for the *positive subring* of S . We say that S is *half-strongly \mathbb{Z}^k -graded* if for every $v \in \mathbb{N}^k$ the multiplication map

$$S_{-v} \otimes_{S_0} S_v \longrightarrow S_0, \quad x \otimes y \mapsto xy \quad (4.1)$$

is surjective.

Theorem 4.2. *Suppose that the \mathbb{N}^k -graded ring $R = S_+$ is the positive subring of a half-strongly \mathbb{Z}^k -graded ring S . Let A be a right ideal in R such that A^r , its right annihilator in R , is non-trivial. Then $A^r \cap R_0 \neq \{0\}$, i.e., A^r contains a non-zero homogeneous element of degree 0.*

Proof. By Theorem 3.1 there exist $v \in \mathbb{N}^k$ and a non-zero element $g \in R_v = S_v$ such that $g \in A^r$. By hypothesis on S the multiplication map in (4.1) is surjective, so there are finitely many elements $x_i \in S_{-v}$ and $y_i \in S_v$ such that $\sum_i x_i y_i = 1$. Now

$$\sum_i (gx_i) \cdot y_i = g \cdot \sum_i x_i y_i = g \neq 0$$

so that there exists an index p with $h := gx_p \neq 0$. By construction $h \in S_0 = R_0$, and $h \in A^r$ as for $a \in A$ we have $ah = (ag)x_p = 0 \cdot x_p = 0$. \square

Theorem 4.2 applies, for example, to a polynomial ring

$$R = K[X_1, X_2, \dots, X_k]$$

with coefficients in a unital ring K ; indeed, we have $R = S_+$ where S is the LAURENT polynomial ring

$$S = K[X_1, X_1^{-1}, X_2, X_2^{-1}, \dots, X_k, X_k^{-1}]$$

(equipped with the usual \mathbb{Z}^k -grading, giving the indeterminate X_j degree e_j , the j th unit vector). For this case, the result has been proved by MCCOY [McC57, Theorem].

5. ANNIHILATORS IN STRONGLY \mathbb{Z}^k -GRADED RINGS

Recall that a \mathbb{Z}^k -graded ring S is called *strongly graded* if the multiplication map (4.1) is surjective for all $v \in \mathbb{Z}^k$.

Theorem 5.1. *Let S be a strongly \mathbb{Z}^k -graded ring, and let $R = S_+$ be its positive subring. Let $f_1, f_2, \dots, f_\ell \in S$, and suppose that there exists a non-zero element $g \in S$ such that*

$$f_j r g = 0 \quad \text{for all } j \text{ and all } r \in R. \quad (5.2)$$

Then there exists a non-zero element $h \in R_0$, homogeneous of degree 0, such that $f_j r h = 0$ for all j and all $r \in R$.

Proof. Without loss of generality we may assume that $g \in R$. Indeed, we may choose $w \in \mathbb{N}^k$ (with sufficiently large positive entries) such that the following holds: if a non-zero homogeneous component of g has degree u then $u + w \in \mathbb{N}^k$. Now the multiplication map in (4.1) (for $v = -w$) is surjective, by hypothesis on S , so there are finitely many elements $y_i \in S_w$ and $x_i \in S_{-w}$ such that $\sum_i y_i x_i = 1$. Since

$$\sum_i (gy_i) \cdot x_i = g \cdot \sum_i y_i x_i = g \neq 0$$

there exists an index p with $gy_p \neq 0$; by choice of w we have $gy_p \in R$, and clearly gy_p satisfies condition (5.2) when used instead of g .

Next, we may choose a $v \in \mathbb{N}^k$ such that the following holds: if a non-zero homogeneous component of one of the f_j has degree u then $u + v \in \mathbb{N}^k$. Now the multiplication map in (4.1) is surjective, by hypothesis on S , so there are finitely many elements $x_i \in S_{-v}$ and $y_i \in S_v$ such that $\sum_i x_i y_i = 1$. Then $y_i f_j \in R$ by choice of v , and for all indices i and j , and all $r \in R$, we have

$$(y_i f_j) r g = y_i (f_j r g) = y_i \cdot 0 = 0$$

by our hypotheses on g and the f_j . This means that the ideal $A = \langle \{y_i f_j\} \rangle$ of R generated by the elements $y_i f_j$ has a non-trivial right annihilator in R . By Theorem 4.2 there exists a non-zero element $h \in R_0$ annihilating A from the right. In particular, $y_i f_j r h = 0$ for all indices i and j , and all $r \in R$. But then we also have the equality

$$f_j r h = \sum_i x_i (y_i f_j r h) = 0$$

for all j and all $r \in R$, proving the Theorem. \square

Corollary 5.3. *Let S be a strongly \mathbb{Z}^k -graded ring. Suppose that S is semi-commutative so that $ab = 0$ implies $asb = 0$ for all $s \in S$. Let $f_1, f_2, \dots, f_\ell \in S$, and suppose that there exists a non-zero element $g \in S$ such that $f_j g = 0$ for all j . Then there exists a non-zero element $h \in S_0$, homogeneous of degree 0, such that $f_j h = 0$ for all j .*

Proof. By definition of semi-commutativity the condition $f_j g = 0$ implies $f_j s g = 0$ for all $s \in S$. In particular, condition (5.2) is satisfied, so Theorem 5.1 applies. \square

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